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Generalization of multi-specializations and multi-asymptotics

By

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Abstract

The purpose of this paper is to report on a new description of geometry appearing in the multi-specialization along a general family of submanifolds and to extend the multi-specialization. The notion of multi-asymptotic expansions is also extended. Locally we can construct new sheaves of multi-asymptotically developable functions closely related with asymptotics along a subvariety with a simple singularity such as a cusp.

§ 1. Introduction

Asymptotically developable expansions of holomorphic functions on a sector are an important tool to study ordinary differential equations with irregular singularities.

Their functorial nature was proven by V. Colin in [1] thanks to formal specialization and more recently in [9] by specializing the (subanalytic) sheaf of Whitney functions.

In higher dimension H. Majima introduced in [7] the notion of strongly asymptotically developable functions along a normal crossing divisor. These functions are related with Whitney holomorphic functions on a multi-sector, as proven in [2].

A natural question arises: can we construct functorially Majima's asymptotics? A positive answer was given in [3], thanks to the multi-specialization applied to Whitney holomorphic functions.

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The aim of this paper is to give a new description of geometry appearing in the multi-specialization along a general family of submanifolds and to extend the multi-specialization. The notion of multi-asymptotic expansions is also extended. Locally we can construct new sheaves of multi-asymptotically developable functions closely related with asymptotics along a subvariety with a simple singularity such as a cusp. The results are extracted from [4].

§ 2. Sheaves on a subanalytic site

The results of this section are extracted from [6] (see also [8] for a more detailed study).

Let X be a real analytic manifold and let k be a field. Denote by $\mathrm{Op}(X_{sa})$ the category of open subanalytic subsets of X. One endows $\mathrm{Op}(X_{sa})$ with the following topology: $S \subset \mathrm{Op}(X_{sa})$ is a covering of $U \in \mathrm{Op}(X_{sa})$ if for any compact K of X there exists a finite subset $S_0 \subset S$ such that $K \cap \bigcup_{V \in S_0} V = K \cap U$. We will call X_{sa} the subanalytic site.

Let $\mathrm{Mod}(k_{X_{sa}})$ denote the category of sheaves on X_{sa} and let $\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$ be the Abelian category of $\mathbb{R}$-constructible sheaves on X.

We denote by $\rho : X \to X_{sa}$ the natural morphism of sites. We have functors

$$\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X) \subset \mathrm{Mod}(k_X) \underset{\rho^{-1}}{\overset{\rho_*}{\rightleftarrows}} \mathrm{Mod}(k_{X_{sa}}).$$

The functors ρ^{-1} and ρ_* are the functors of inverse image and direct image associated to ρ. The functor ρ^{-1} admits a left adjoint, denoted by $\rho_!$. The sheaf $\rho_! F$ is the sheaf associated to the presheaf $\mathrm{Op}(X_{sa}) \ni U \mapsto F(\overline{U})$.

The functor ρ_* is fully faithful and exact on $\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$ and we identify $\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$ with its image in $\mathrm{Mod}(k_{X_{sa}})$ by ρ_*.

Let X, Y be two real analytic manifolds, and let $f : X \to Y$ be a real analytic map. We get the internal operations $\mathcal{H}om$, $\otimes$ and the external operations f^{-1} and f_*, which are always defined for sheaves on Grothendieck topologies. For subanalytic sheaves we can also define the functor of proper direct image $f_{!!}$. The notation $f_{!!}$ is due to the fact that $f_{!!} \circ \rho_* \not\simeq \rho_* \circ f_!$ in general. While the functors f^{-1} and $\otimes$ are exact, the functors $\mathcal{H}om$, f_* and $f_{!!}$ are left exact and admit right derived functors. The functor $Rf_{!!}$ admits a right adjoint, denoted by $f^!$, and we get the usual isomorphisms like projection formula, base change formula, Künneth formula.

§ 3. Multi-normal deformation

We refer to [5] for the definition of the classical normal deformation. For simplicity, we assume $X = \mathbb{C}^n$, with coordinates $z = (z_1, \ldots, z_n)$. Let $\chi = \{M_1, \ldots, M_\ell\}$ be a family of submanifolds, $M_j = \{z_i = 0,\, i \in I_j\}$, $I_j \subseteq \{1, \ldots, n\}$. We associate to χ an action $\mu_j(z, \lambda) = (\lambda^{a_{j1}} z_1, \ldots, \lambda^{a_{jn}} z_n)$ with $a_{ji} \in \mathbb{N}_0$ ($\mathbb{N}_0 := \mathbb{N} \cup \{0\}$), we call A_χ the matrix (a_{ji}) associated to the action.

Let $A_\chi = (a_{ji})$ be an $\ell \times n$ matrix with $a_{ji} \in \mathbb{N}_0$, $a_{ji} \neq 0$ if $i \in I_j$, $a_{ji} = 0$ otherwise. We can define a general normal deformation $\widetilde{X} = \mathbb{C}^n \times \mathbb{C}^\ell$ with the map $p : \widetilde{X} \to X$ defined by

$$p(x, t) = (\varphi_1(t) x_1, \ldots, \varphi_n(t) x_n)$$

with

$$\varphi_i(t) = \prod_{j=1}^{\ell} t_j^{a_{ji}} \qquad (i = 1, 2, \ldots, n). \tag{3.1}$$

Comparing with the matrix A_χ, when $t \in (\mathbb{R}^+)^\ell$ we have

$$(\log \varphi_1, \ldots, \log \varphi_n) = (\log t_1, \ldots, \log t_\ell) A_\chi.$$

Set $S_\chi = \{t_1 = \cdots = t_\ell = 0\}$. Let $s : S_\chi \hookrightarrow \widetilde{X}$ be the inclusion, $\Omega = \{t_1, \ldots, t_\ell > 0\}$, $M = \bigcap_{i=1}^{\ell} M_i$. We get a commutative diagram

$$\begin{tikzcd} S \arrow[r, "s"] \arrow[d, "\tau"] & \widetilde{X} \arrow[d, "p"] & \Omega \arrow[l, "i_\Omega"'] \arrow[dl, "\widetilde{p}"] \\ M \arrow[r, "i"] & X. & \end{tikzcd}$$

For simplicity we assume that $\ell \leq n$ and the $\ell \times \ell$ submatrix $A_{\chi\ell}$ made from the first ℓ-columns and the first ℓ-rows in A_χ is invertible (for the cases without these assumptions, see [4]). We are interested in the zero section S_χ of $\widetilde{X}$ defined by $\{t_i = 0,\, i = 1, \ldots, \ell\}$. In particular (for simplicity) points $\xi = (\xi_1, \ldots, \xi_n)$, $\xi_i \neq 0$, $i = 1, \ldots, \ell$.

Example 3.1. Let us consider some examples in $\mathbb{C}^2$.

(Majima) Let $X = \mathbb{C}^2 = (z_1, z_2)$, $M_i = \{z_i = 0\}$, $i = 1, 2$. Consider the matrix

$$A_\chi = \begin{pmatrix} 1\,0 \\ 0\,1 \end{pmatrix}, \quad \varphi_1 = t_1, \quad \varphi_2 = t_2.$$

We have $\widetilde{X} = (z_1, z_2, t_1, t_2)$, $p : (z_1, z_2, t_1, t_2) \to (z_1 t_1, z_2 t_2)$.

(Takeuchi) Let $X = \mathbb{C}^2 = (z_1, z_2)$, $M_1 = \{0\}$, $M_2 = \{z_2 = 0\}$. Consider the matrix

$$A_{\mathcal{X}} = \begin{pmatrix} 1\,1 \\ 0\,1 \end{pmatrix}, \quad \varphi_1 = t_1, \quad \varphi_2 = t_1 t_2.$$

We have $\widetilde{X} = (z_1, z_2, t_1, t_2)$, $p : (z_1, z_2, t_1, t_2) \to (z_1 t_1, z_2 t_1 t_2)$. This is the binormal deformation of [10].

(Cusp) Let $X = \mathbb{C}^2 = (z_1, z_2)$, $M_1 = M_2 = \{0\}$. Consider the matrix

$$A_{\mathcal{X}} = \begin{pmatrix} 3\,2 \\ 1\,1 \end{pmatrix}, \quad \varphi_1 = t_1^3 t_2, \quad \varphi_2 = t_1^2 t_2$$

We have $\widetilde{X} = (z_1, z_2, t_1, t_2)$, $p : (z_1, z_2, t_1, t_2) \to (z_1 t_1^3 t_2, z_2 t_1^2 t_2)$.

§ 4. Multi-sectors

Let $\xi = (\xi_1, \dots, \xi_n) \in S_{\mathcal{X}}$ with $\xi_i \neq 0$, $i = 1, \dots, \ell$. Let $\epsilon > 0$, and let $W = W_1 \times \cdots \times W_n$, W_i open conic cone in $\mathbb{C}$ containing the direction ξ_i. Set $|z|_\ell = (|z_1|, \dots, |z_\ell|)$. A multi-sector $S(W, \epsilon)$ is an element of the family $C(\xi)$ defined as follows:

$$S(W,\epsilon) = \left\{ (z_1, \dots, z_n);\ \begin{matrix} z_i \in W_i \qquad (i = 1, \dots, n), \\ \varphi_i^{-1}(|z|_\ell) < \epsilon \qquad (i \leq \ell), \\ |\xi_i| - \epsilon < \dfrac{|z_i|}{\varphi_i(\varphi^{-1}(|z|_\ell))} < |\xi_i| + \epsilon \quad (i > \ell) \end{matrix} \right\},$$

where $\epsilon > 0$, and W_i are cones in $\mathbb{C}$ containing the direction ξ_i and φ_i^{-1} is such that $\varphi_i(\varphi^{-1}(z)) = z_i$, $i = 1, \dots, \ell$. Comparing with the matrix $A_{\mathcal{X}}$, when $t \in (\mathbb{R}^+)^\ell$ we have

$$(\log \varphi_1^{-1}, \dots, \log \varphi_\ell^{-1}) = (\log t_1, \dots, \log t_\ell) A_{\mathcal{X}}^{-1}.$$

We say that $S(W', \epsilon') < S(W, \epsilon)$ ($S(W', \epsilon')$ is properly contained in $S(W, \epsilon)$) if $\overline{W'} \backslash \{0\} \subset W$ and $\epsilon' < \epsilon$. The main geometrical properties of a multi-sector $S := S(W, \epsilon)$ are the following:

- S is locally cohomologically trivial. That is, $R\mathcal{H}om(\mathbb{C}_S; \mathbb{C}_X) = \mathbb{C}_{\overline{S}}$,
- S is 1-regular, that is, there exists a constant $C > 0$ satisfying that, for any point p and q in S , there exists a rectifiable curve in V which joins p and q and whose length is $\leq C|p - q|$.

Example 4.1. Let us consider some examples in $\mathbb{C}^2$.

(Majima) Let $X = \mathbb{C}^2 = (z_1, z_2)$, $M_i = \{z_i = 0\}$, $i = 1, 2$. Then

$$A_\chi^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \varphi_1^{-1} = t_1, \quad \varphi_2^{-1} = t_2.$$

We have

$$C(\xi) \ni S(W, \epsilon) = \left\{ z \in X; \begin{array}{ll} z_i \in W_i & (i = 1, 2), \\ |z_i| < \epsilon & (i = 1, 2) \end{array} \right\},$$

where $\epsilon > 0$ and W_i conic open subset containing ξ_i.

(Takeuchi) Let $X = \mathbb{C}^2 = (z_1, z_2)$, $M_1 = \{0\}$, $M_2 = \{z_2 = 0\}$. Then

$$A_\chi^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \varphi_1^{-1} = t_1, \quad \varphi_2^{-1} = \frac{t_2}{t_1}.$$

We have

$$C(\xi) \ni S(W, \epsilon) = \left\{ z \in X; \begin{array}{l} z_i \in W_i \quad (i = 1, 2), \\ |z_1| < \epsilon \\ |z_2| < \epsilon |z_1| \end{array} \right\},$$

where $\epsilon > 0$ and W_i a conic open subset containing ξ_i. These are the multi-sectors of [10].

(Cusp) Let $X = \mathbb{C}^2 = (z_1, z_2)$, $M_1 = M_2 = \{0\}$. Then

$$A_\chi^{-1} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}, \quad \varphi_1^{-1} = \frac{t_1}{t_2}, \quad \varphi_2^{-1} = \frac{t_2^3}{t_1^2}.$$

We have

$$C(\xi) \ni S(W, \epsilon) = \left\{ z \in X; \begin{array}{l} z_i \in W_i \quad (i = 1, 2), \\ |z_1| < \epsilon |z_2| \\ |z_2|^3 < \epsilon |z_1|^2 \end{array} \right\},$$

where $\epsilon > 0$ and W_i a conic open subset containing ξ_i.

§ 5. Multi-specialization

The multi-specialization along χ is the functor

$$\begin{aligned} \nu_\chi : D^b(\mathbb{C}_{X_{sa}}) &\to D^b(\mathbb{C}_{S_\chi}) \\ F &\mapsto \rho^{-1} s^{-1} R\Gamma_\Omega p^{-1} F. \end{aligned}$$

where $\rho : S_{\chi sa} \to S_\chi$ is the natural functor of sites (here D^b denotes the bounded derived category of sheaves). Thanks to the functor $\rho^{-1} : D^b(\mathbb{C}_{S_{sa}}) \to D^b(\mathbb{C}_S)$ we can calculate the fibers at $\xi \in S_\chi$ which are given by

$$(H^j \nu_\chi F)_\xi \simeq \varinjlim_{S(W,\epsilon)} H^j(S(W,\epsilon); F),$$

where $S(W,\epsilon)$ ranges through the family $C(\xi)$.

Let $\mathcal{O}_X^{\mathrm{w}} \in D^b(\mathbb{C}_{X_{sa}})$ denote the subanalytic sheaf of Whitney holomorphic functions. The sheaf of multi-asymptotically developable holomorphic functions is the multi-specialization of $\mathcal{O}_X^{\mathrm{w}}$:

$$\nu_\chi \mathcal{O}_X^{\mathrm{w}}.$$

§ 6. Multi-asymptotics

Let $\mathcal{P}_\ell$ be the set of nonempty subsets of $\{1,\dots,\ell\}$. Let $J \in \mathcal{P}_\ell$. We use the following notations:

- $I_J = \bigcup_{j\in J} I_j$,
- $M_J = \bigcap_{j\in J} M_j$,
- $z_J = (z_i)_{i\in I_J}$, $z_J^C = (z_i)_{i\notin I_J}$,
- $\mathbb{N}_0^J = \{(\alpha_1,\dots,\alpha_n) \in \mathbb{N}_0^n,\ \alpha_i = 0\ i \notin I_J\}$
- $\pi_J : X \to M_J$ the projection,
- given $S \subset X$, $S_J = \pi_J(S)$.

Let $S := S(W,\epsilon)$ be a multi-sector. We say that $F = \{F_J\}_{J\in\mathcal{P}_\ell}$ is a total family of coefficients of multi-asymptotic expansion along χ on S if each F_J consists of a family $\{f_{J,\alpha}\}_{\alpha\in\mathbb{N}_0^J}$ of holomorphic functions on S_J.

Given a total family of coefficients $F = \{F_J\}_{J\in\mathcal{P}_\ell}$ and $N = (n_1,\dots,n_\ell) \in \mathbb{N}_0^\ell$, the approximate function of degree N of F is

$$\mathrm{App}^{<N}(F;z) = \sum_{J\in\mathcal{P}_\ell} (-1)^{\sharp J+1} \sum_{\alpha\in A_J(N)} \frac{f_{J,\alpha}(z_J^C)}{\alpha!} z^\alpha,$$

where

$$A_J(N) = \left\{ \alpha \in \mathbb{N}_0^J; \sum_{i \in I_j} a_{ji}\alpha_i < n_j \text{ for any } j \in J \right\}.$$

(i.e., $\alpha\cdot$ (j-th line of A_χ) $< n_j$).

We say that f is multi-asymptotically developable to $F = \{F_J\}$ along χ on $S = S(W, \epsilon)$ if and only if for any cone $S' = S(W', \epsilon')$ properly contained in S and for any $N = (n_1, \ldots, n_\ell) \in \mathbb{N}_0^\ell$, there exists a constant C such that

$$\left| f(z) - \mathrm{App}^{<N}(F; z) \right| \leq C \prod_{1 \leq j \leq \ell} \varphi_j^{-1}(|z|_\ell)^{n_j} \qquad (z \in S').$$

Example 6.1. Let us consider some examples in $\mathbb{C}^2$.

(Majima) Let $M_{\{1\}} = \{z_1 = 0\}$, $M_{\{2\}} = \{z_2 = 0\}$, $M_{\{1,2\}} = \{0\}$,

$$S(W, \epsilon) = \left\{ z \in X; \begin{array}{ll} z_i \in W_i & (i = 1, 2), \\ |z| < \epsilon & \end{array} \right\},$$

where the norm $|z|$ denotes $\max\{|z_1|, |z_2|\}$. We have $S_{\{1,2\}} = \{\mathrm{pt}\}$ and

$$S_{\{1\}} = \left\{ z \in M_1; \begin{array}{l} z_2 \in W_2, \\ |z| < \epsilon \end{array} \right\}, \quad S_{\{2\}} = \left\{ z \in M_2; \begin{array}{l} z_1 \in W_1, \\ |z| < \epsilon \end{array} \right\}.$$

A total family of coefficients is

$$F = \left\{ \{f_{\{1\},\alpha_1}(z_2)\}_{\alpha_1 \in \mathbb{N}_0}, \{f_{\{2\},\alpha_2}(z_1)\}_{\alpha_2 \in \mathbb{N}_0}, \{f_{\{1,2\},\alpha}\}_{\alpha \in \mathbb{N}_0^2} \right\},$$

where $f_{\{1\},\alpha_1}(z_2)$ (resp. $f_{\{2\},\alpha_2}(z_1)$) is holomorphic in $S_{\{1\}}$ (resp. $S_{\{2\}}$) and $f_{\{1,2\},\alpha} \in \mathbb{C}$. Let

$$A_\chi = \begin{pmatrix} 1\,0 \\ 0\,1 \end{pmatrix}, \qquad N = (n_1, n_2) \in \mathbb{N}_0^2.$$

We have

$$\begin{aligned} A_{\{1\}}(N) &= \{\alpha_1 \in \mathbb{N}_0,\ \alpha_1 < n_1\}, \\ A_{\{2\}}(N) &= \{\alpha_2 \in \mathbb{N}_0,\ \alpha_2 < n_2\}, \\ A_{\{1,2\}}(N) &= \{(\alpha_1, \alpha_2) \in \mathbb{N}_0^2,\ \alpha_1 < n_1,\ \alpha_2 < n_2\} \end{aligned}$$

and

$$\begin{aligned} \mathrm{App}^{<N}(F; z) = &\sum_{\alpha_1 < n_1} f_{\{1\},\alpha_1}(z_2) \frac{z_1^{\alpha_1}}{\alpha_1!} + \sum_{\alpha_2 < n_2} f_{\{2\},\alpha_2}(z_1) \frac{z_2^{\alpha_2}}{\alpha_2!} \\ &- \sum_{\substack{\alpha_1 < n_1 \\ \alpha_2 < n_2}} f_{\{1,2\},\alpha} \frac{z_1^{\alpha_1} z_2^{\alpha_2}}{\alpha_1! \alpha_2!}. \end{aligned}$$

A holomorphic function f is strongly asymptotically developable if, for any multi-sector S' properly contained in $S(W,\epsilon)$ and for any $N=(n_1,n_2)\in\mathbb{N}_0^2$, there exists a positive constant $C_{S',N}$ such that

$$\left|f(z)-\mathrm{App}^{<N}(F;z)\right|\leq C_{S',N}|z_1|^{n_1}|z_2|^{n_2}$$

with $z\in S'$. This corresponds to Majima's asymptotics of [7].

(Takeuchi) Let $M_{\{1\}}=\{0\}$, $M_{\{2\}}=\{z_2=0\}$, $M_{\{1,2\}}=\{0\}$,

$$S(W,\epsilon)=\left\{z\in X;\ \begin{array}{l} z_i\in W_i\quad (i=1,2),\\ |z_1|<\epsilon,\\ |z_2|<\epsilon|z_1|\end{array}\right\}.$$

We have $S_{\{1\}}=S_{\{1,2\}}=\{\mathrm{pt}\}$ and

$$S_{\{2\}}=\left\{z\in M_2;\ \begin{array}{l} z_1\in W_1,\\ |z|<\epsilon\end{array}\right\}.$$

A total family of coefficients is

$$F=\left\{\{f_{\{1\},\alpha}\}_{\alpha\in\mathbb{N}_0^2},\{f_{\{2\},\alpha_2}(z_1)\}_{\alpha_2\in\mathbb{N}_0},\{f_{\{1,2\},\alpha}\}_{\alpha\in\mathbb{N}_0^2}\right\},$$

where $f_{\{2\},\alpha_2}(z_1)$ is holomorphic in $S_{\{2\}}$ and $f_{\{1\},\alpha},f_{\{1,2\},\alpha}\in\mathbb{C}$. Let

$$A_\chi=\begin{pmatrix}1\,1\\0\,1\end{pmatrix},\qquad N=(n_1,n_2)\in\mathbb{N}_0^2.$$

We have

$$\begin{aligned}
A_{\{1\}}(N)&=\{(\alpha_1,\alpha_2)\in\mathbb{N}_0^2,\ \alpha_1+\alpha_2<n_1\},\\
A_{\{2\}}(N)&=\{\alpha_2\in\mathbb{N}_0,\ \alpha_2<n_2\},\\
A_{\{1,2\}}(N)&=\{(\alpha_1,\alpha_2)\in\mathbb{N}_0^2,\ \alpha_1+\alpha_2<n_1,\ \alpha_2<n_2\}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{App}^{<N}(F;z)=&\sum_{\alpha_1+\alpha_2<n_1} f_{\{1\},\alpha}\frac{z_1^{\alpha_1}z_2^{\alpha_2}}{\alpha_1!\alpha_2!}+\sum_{\alpha_2<n_2} f_{\{2\},\alpha_2}(z_1)\frac{z_2^{\alpha_2}}{\alpha_2!}\\
&-\sum_{\substack{\alpha_1+\alpha_2<n_1\\ \alpha_2<n_2}} f_{\{1,2\},\alpha}\frac{z_1^{\alpha_1}z_2^{\alpha_2}}{\alpha_1!\alpha_2!}.
\end{aligned}$$

A holomorphic function f is strongly asymptotically developable if, for any multi-sector S' properly contained in $S(W,\epsilon)$ and for any $N=(n_1,n_2)\in\mathbb{N}_0^2$, there exists a positive constant $C_{S',N}$ such that

$$\left|f(z)-\mathrm{App}^{<N}(F;z)\right|\leq C_{S',N}|z_1|^{n_1-n_2}|z_2|^{n_2}$$

with $z \in S'$. This corresponds to Takeuchi's asymptotics of [3].

(Cusp) Let $M_{\{1\}} = M_{\{2\}} = M_{\{1,2\}} = \{0\}$,

$$S(W,\epsilon) = \left\{ \begin{array}{c} z_i \in W_i \quad (i=1,2), \\ z \in X;\ |z_1| < \epsilon |z_2|, \\ |z_2|^3 < \epsilon |z_1|^2 \end{array} \right\}.$$

We have $S_{\{1\}} = S_{\{2\}} = S_{\{1,2\}} = \{0\}$. A total family of coefficients is given by

$$F = \left\{ \{f_{\{1\},\alpha}\}_{\alpha\in\mathbb{N}_0^2}, \{f_{\{2\},\alpha}\}_{\alpha\in\mathbb{N}_0^2}, \{f_{\{1,2\},\alpha}\}_{\alpha\in\mathbb{N}_0^2} \right\},$$

where $f_{\{1\},\alpha}, f_{\{2\},\alpha}, f_{\{1,2\},\alpha} \in \mathbb{C}$. Let

$$A_\chi = \begin{pmatrix} 3\,2 \\ 1\,1 \end{pmatrix}, \qquad N = (n_1, n_2) \in \mathbb{N}_0^2.$$

We have

$$\begin{aligned} A_{\{1\}}(N) &= \{(\alpha_1,\alpha_2) \in \mathbb{N}_0^2,\ 3\alpha_1 + 2\alpha_2 < n_1\}, \\ A_{\{2\}}(N) &= \{(\alpha_1,\alpha_2) \in \mathbb{N}_0^2,\ \alpha_1 + \alpha_2 < n_2\}, \\ A_{\{1,2\}}(N) &= \{(\alpha_1,\alpha_2) \in \mathbb{N}_0^2,\ 3\alpha_1 + 2\alpha_2 < n_1,\ \alpha_1 + \alpha_2 < n_2\} \end{aligned}$$

and

$$\begin{aligned} \mathrm{App}^{<N}(F;z) = & \sum_{3\alpha_1+2\alpha_2<n_1} f_{\{1\},\alpha} \frac{z_1^{\alpha_1} z_2^{\alpha_2}}{\alpha_1!\alpha_2!} + \sum_{\alpha_1+\alpha_2<n_2} f_{\{2\},\alpha} \frac{z_1^{\alpha_1} z_2^{\alpha_2}}{\alpha_1!\alpha_2!} \\ & - \sum_{\substack{3\alpha_1+2\alpha_2<n_1 \\ \alpha_1+\alpha_2<n_2}} f_{\{1,2\},\alpha} \frac{z_1^{\alpha_1} z_2^{\alpha_2}}{\alpha_1!\alpha_2!}. \end{aligned}$$

A holomorphic function f is strongly asymptotically developable if, for any multi-sector S' properly contained in $S(W,\epsilon)$ and for any $N = (n_1,n_2) \in \mathbb{N}_0^2$, there exists a positive constant $C_{S',N}$ such that

$$\left| f(z) - \mathrm{App}^{<N}(F;z) \right| \leq C_{S',N} |z_1|^{n_1 - 2n_2} |z_2|^{3n_2 - n_1}$$

with $z \in S'$.

§ 7. Multi-specialization and multi-asymptotics

One can check that multi-asymptotics on a multi-sector S are Whitney on $S' < S$. Moreover the geometrical properties of a multi-sector imply vanishing of the cohomology of multi-specialization. Combining these two results we have that the sheaf $\nu_\chi \mathcal{O}_X^{\mathrm{w}}$ is concentrated in degree zero and we have

$$\nu_\chi \mathcal{O}_X^{\mathrm{w}}(S) = \{f \text{ holomorphic and multi-asymptotically developable on } S\}.$$

Set $Z := \cup_{j=1}^{\ell} M_j$. Let $\mathcal{O}_X^{\mathrm{w}}$, $\mathcal{O}_{X|X\setminus Z}^{\mathrm{w}}$, $\mathcal{O}_{X|Z}^{\mathrm{w}}$ denote the sheaves on the subanalytic site X_{sa} of Whitney holomorphic functions, flat Whitney holomorphic functions and Whitney holomorphic functions on Z respectively. See [3] for more details.

We can prove functorially the exactness of the sequence

$$0 \to \nu_\chi \mathcal{O}_{X|X\setminus Z}^{\mathrm{w}} \to \nu_\chi \mathcal{O}_X^{\mathrm{w}} \to \nu_\chi \mathcal{O}_{X|Z}^{\mathrm{w}} \to 0. \tag{7.1}$$

In the case of Majima's asymptotics we have the isomorphisms (outside the zero section)

$$\begin{aligned} \mathcal{A}_X &\xrightarrow{\sim} \nu_\chi \mathcal{O}_X^{\mathrm{w}}, \\ \mathcal{A}_X^{<0} &\xrightarrow{\sim} \nu_\chi \mathcal{O}_{X|X\setminus Z}^{\mathrm{w}}, \\ \mathcal{A}_X^{CF} &\xrightarrow{\sim} \nu_\chi \mathcal{O}_{X|Z}^{\mathrm{w}}, \end{aligned}$$

where as usual, we denote by $\mathcal{A}_X$, $\mathcal{A}_X^{<0}$, $\mathcal{A}_X^{CF}$ the sheaves of strongly asymptotically developable functions, flat asymptotics and consistent families of coefficients. In this case (7.1) is the Borel-Ritt exact sequence for Majima's asymptotics.

So we have obtained a general Borel-Ritt exact sequence for multi-asymptotically developable functions.

Example 7.1. We end this paper with some examples of consistent families in $\mathbb{C}^2$. For the general definition we refer to [4].

(Majima) The family

$$F = \{\{f_{\{1\},\alpha_1}(z_2)\}, \{f_{\{2\},\alpha_2}(z_1)\}, \{f_{\{1,2\},(\alpha_1,\alpha_2)}\}\}$$

is consistent if

- $f_{\{1\},\alpha_1}(z_2)$ is strongly asymptotically developable to
$$\{f_{\{1,2\},(\alpha_1,\alpha_2)}\}_{\alpha_2 \in \mathbb{N}_0}$$
on $S_{\{1\}}$ for each $\alpha_1 \in \mathbb{N}_0$,
- $f_{\{2\},\alpha_2}(z_1)$ is strongly asymptotically developable to
$$\{f_{\{1,2\},(\alpha_1,\alpha_2)}\}_{\alpha_1 \in \mathbb{N}_0}$$
on $S_{\{2\}}$ for each $\alpha_2 \in \mathbb{N}_0$.

(Takeuchi) The family

$$F = \{\{f_{\{1\},(\alpha_1,\alpha_2)}\}, \{f_{\{2\},\alpha_2}(z_1)\}, \{f_{\{1,2\},(\alpha_1,\alpha_2)}\}\}$$

is consistent if

- $f_{\{1\},(\alpha_1,\alpha_2)} = f_{\{1,2\},(\alpha_1,\alpha_2)}$ for each $(\alpha_1, \alpha_2) \in \mathbb{N}_0^2$,
- $f_{\{2\},\alpha_2}(z_1)$ is strongly asymptotically developable to

$$\{f_{\{1,2\},(\alpha_1,\alpha_2)}\}_{\alpha_1 \in \mathbb{N}_0}$$

  on $S_{\{2\}}$ for each $\alpha_2 \in \mathbb{N}_0$.

(Cusp) The family

$$F = \{\{f_{\{1\},(\alpha_1,\alpha_2)}\}, \{f_{\{2\},(\alpha_1,\alpha_2)}\}, \{f_{\{1,2\},(\alpha_1,\alpha_2)}\}\}$$

is consistent if

- $f_{\{1\},(\alpha_1,\alpha_2)} = f_{\{2\},(\alpha_1,\alpha_2)} = f_{\{1,2\},(\alpha_1,\alpha_2)}$ for each $(\alpha_1, \alpha_2) \in \mathbb{N}_0^2$.

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